RESEARCH REPORT

SOME 2-(2n+1,n,n-1) DESIGNS WITH MULTIPLE EXTENSIONS

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ABSTRACT

A $2-(2n+1,n,\lambda)$ design can always be extended to a $3-(2n+2,n+1,\lambda)$ design by complementation. If λ is large enough there may be other methods of extension. By constructing non-self-complementary 3-(18,9,7) designs it is shown that there is a 2-(17,8,7) design with 16 extensions. The method generalises to construct non-self complementary 3-(2n+2,n+1,n-1) designs for larger values of n.

A t-(v,k,λ) design on v varieties (or points or symbols) consists of k-sets, called blocks, chosen from the v varieties in such a way that all the blocks are different and each unordered t-tuple of varieties occurs exactly λ times in the design. In this paper structures resembling t-designs but not having all the required properties will be called arrays. The meaning of the word 'array' each time it is used should be clear from its immediate context

In a t-design let λ_i, $(0 \leqslant i \leqslant t)$, be the number of times each unordered i-tuple occurs; in particular $\lambda_0 = b$, the number of blocks, and $\lambda_1 = r$, the number of replications of each variety. Then the λ_i's are given by the standard equations

$$\lambda_i = \frac{(v-i)(v-i-1)(v-i-2)\ldots(v-t+1)}{(k-i)(k-i-1)(k-i-2)\ldots(k-t+1)}\,\lambda$$

with of course $\lambda_t = \lambda$. If the design contains all possible k-sets from v varieties then it is said to be trivial.

From a given t-design D a (t-1)-design can be constructed by discarding all blocks not containing a given variety x which is then deleted from the blocks that remain. The design so obtained is called a restriction (or contraction) of D on x and is denoted by D_x. The reverse process can sometimes be performed by adding a new variety x to all the blocks in a (t-1)-design to form new blocks which are supplemented by further new blocks not containing x to make a t-design. In this case the new design is called an extension of the original design.

For 2-$(2n+1,n,\lambda)$ designs there are two well-known results about extensions to 3-designs. They are:

(i) any $2-(2n+1,n,\lambda)$ design can be extended to a $3-(2n+2,n+1,\lambda)$ design by complementation;

(ii) for $2\lambda = n-1$ (i.e. for Hadamard designs) there is only one way of extending to a 3-design and that is by complementation.

In an extension by complementation each original block has the same new variety added to it. Then complements with respect to the extended variety set are taken to form further new blocks. The resulting design is self-complementary and each of its blocks contains half the total number of varieties.

I have always treated (i) and (ii) as folk theorems but I am sure they are not and I would like to know who the originators are so they can be given their rightful credit. The proofs of both are worth reproducing here since not only are they brief but they also illustrate basic counting principles. To prove (i) let N_3 be the number of blocks in the $2-(2n+1,n,\lambda)$ design which contain all three of a given triple of varieties. Let N_0 be the number of blocks containing none of them. Then by the principle of inclusion and exclusion

$$N_0 = \binom{3}{0}b - \binom{3}{1}r + \binom{3}{2}\lambda - \binom{3}{3}N_3$$

which yields $N_0 + N_3 = \lambda$. Therefore in the array obtained by complementation there are λ complementary pairs of blocks continuing a given triple and so the array is a 3-design.

The proof of (ii) uses a property of symmetric 2-designs (the designs for which $b = v$) namely, any block in such a design intersects any other in exactly λ varieties. Now in a $3-(2n+2,n+1,\frac{n}{2}-\frac{1}{2})$ design if two blocks A and B have a variety in common then a restriciton on that variety leads to a symmetric 2-design in which any pair of blocks have $\frac{n}{2}-\frac{1}{2}$ varieties in common. Therefore in the 3-design A and B either intersect in $\frac{n}{2}+\frac{1}{2}$ varieties or are disjoint in which case they are complementary. Now take a block A and let N be the number of blocks intersecting it. Then count the pairs (x,Y) where x is a variety in the block Y and also in A with $Y \neq A$. Then counting these pairs in two ways one has

$$\frac{1}{2} N(n+1) = (n+1)(r-1)$$

where r is the replication number for the 3-design and is equal to $(2n+1)$. Thus $N = 4n$. But the total number of blocks b is $(4n+2)$. Of these one is A and 4n are blocks intersecting A so there must be one block not intersecting A. Hence a $3-(2n+2,n+1,\frac{n}{2}-\frac{1}{2})$ design is always self-complementary.

Among the $2-(2n+1,n,\lambda)$ designs the Hadamard designs are those with the smallest value of λ and their extensions to 3-designs are uniquely determined. For larger values of λ however it may be possible to extend a 3-design by methods other than by complementation in which case non-self-complementary $3-(2n+2,n+1,\lambda)$ designs will be formed. Indeed among the eleven non-isomorphic 2-(9,4,3) designs (Stanton, Mullin and Bate [4]: and also [1] and [3]) there are just two that can be extended in more than one way [1]. One of these has three different extensions two of which are isomorphic. The other one, which is the genesis of this paper, can be extended in just two ways. This is the baby of a whole family of designs with multiple extensions to 3-designs. The non-self-complementary 3-(10,5,3) design that it generates can be presented by using two sets of five symbols which here will be distinguished by using two differing typefaces, **01234** and *01234*. (The reader may find it worthwhile to write the designs in this paper with two different colours of ink, say red and blue.) The design has 36 blocks subdivided into sets of 20 and 16 labelled DD* and H respectively (fig 1).

0	*1*	**1**	*4*	**4**	*0*	*2*	**2**	*3*	**3**
0	*1*	**1**	*4*	**4**	**0**	*2*	**2**	*3*	**3**
1	*2*	**2**	*0*	**0**	*1*	*3*	**3**	*4*	**4**
1	*2*	**2**	*0*	**0**	**1**	*3*	**3**	*4*	**4**
2	*3*	**3**	*1*	**1**	*2*	*4*	**4**	*0*	**0**
2	*3*	**3**	*1*	**1**	**2**	*4*	**4**	*0*	**0**
3	*4*	**4**	*2*	**2**	*3*	*0*	**0**	*1*	**1**
3	*4*	**4**	*2*	**2**	**3**	*0*	**0**	*1*	**1**
4	*0*	**0**	*3*	**3**	*4*	*1*	**1**	*2*	**2**
4	*0*	**0**	*3*	**3**	**4**	*1*	**1**	*2*	**2**

DD* blocks

0	*1*	*2*	*3*	*4*
0	*1*	*2*	**3**	**4**
0	*1*	**2**	*3*	**4**
0	*1*	**2**	**3**	*4*
0	**1**	*2*	*3*	**4**
0	**1**	*2*	**3**	*4*
0	**1**	**2**	*3*	*4*
0	**1**	**2**	**3**	**4**
0	*1*	**2**	**3**	**4**
0	*1*	**2**	*3*	*4*
0	*1*	*2*	**3**	*4*
0	*1*	*2*	*3*	**4**
0	**1**	**2**	**3**	*4*
0	**1**	**2**	*3*	**4**
0	**1**	*2*	**3**	**4**
0	**1**	*2*	*3*	*4*

H blocks

Figure 1: A non-self-complementary 3-(10,5,3) design:

$t = 3,\ v = 10,\ b = 36,\ r = 18,\ \lambda_2 = 8,\ \lambda_3 = 3.$

A restriction on *0* say produces a 2-(9,4,3) design which can then be extended by complementation to make a self-complementary 3-(10,5,3) design. As the DD* blocks can be arranged in complementary pairs this design can be formed from figure 1 by replacing the last 8 blocks in the H section with those in figure 2. Thus we have a 2-(9,4,3) design with two different extensions.

0	**1**	**2**	**3**	**4**
0	**1**	**2**	*3*	*4*
0	**1**	*2*	**3**	*4*
0	**1**	*2*	*3*	**4**
0	*1*	**2**	**3**	*4*
0	*1*	**2**	*3*	**4**
0	*1*	*2*	**3**	**4**
0	*1*	*2*	*3*	*4*

Figure 2: Replace the last 8 blocks of figure 1 by these blocks to make a self-complementary 3-(10,5,3) design.

In figure 1 the DD* blocks contain two sorts of triples, those like $xy\mathbf{z}$ in which there are 3 distinct numerical symbols and those like $x\mathbf{xy}$ in which a number is repeated. Triples like xyz, $xy\mathbf{z}$, $x\mathbf{yz}$ etc. occur just once in the DD* blocks. Triples with a repeated number occur 3 times in the DD* blocks. Therefore since each triple occurs 3 times in the whole design the H blocks cannot contain any pairs $\mathbf{x}x$. Thus each H block contains all five numbers and it is the pattern of type-faces that really matters for these blocks. If the type-faces are replaced by +1 and -1 then the resulting columns (reading down the blocks) are like those that occur in a standardised Hadamard matrix; hence the name H blocks.

Now for the DD* blocks the permutation (πx) is an autormorphism so there is no loss of generality in taking the first H block to contain symbols of all one type. The intention is to form the H blocks to complete a non-self-complementary 3-(10,5,3) design. It can be shown that if two blocks of a 3-(10,5,3) design intersect in four varieties, then their complements are both in the design. With this result as a guide it is not difficult to complete the H blocks in figure 1.

If in the DD* blocks of figure 1 all the symbols of one type are omitted then the resulting blocks can be rearranged as in figure 3 to form a 2-(5,2,1) design D and

1	4	2	3	
2	0	3	4	
3	1	4	0	D: 2-(5,2,1)
4	2	0	1	
0	3	1	2	

0	1	4	0	2	3	
1	2	0	1	3	4	
2	3	1	2	4	0	D*: 2-(5,3,3)
3	4	2	3	0	1	
4	0	3	4	1	2	

Figure 3: A block pair disjoint design D and its complement D* (with flag-poles to the fore).

its complement a 2-(5,3,3) design D*. The design D although trivial has the property that its blocks can be arranged in disjoint pairs so there is a symbol missing from each pair and each symbol is omitted just once from a disjoint pair. Such a design will be called a block pair disjoint design. The complement D* of such a design is formed by adding the missing symbol of each disjoint pair to each block of the pair. In the blocks of D* these symbols occupy special positions which will be called flag-poles. Thus each symbol is a flag-pole twice. The DD* blocks of figure 1 are constructed by a duplication of the numbers of the D* blocks which are not flag-poles followed by an interchange of type-faces to produce further blocks.

Note that, although the circumstances are trivial, the array of figure 3 is a 3-array and would be a 3-design formed by complementation if an extra symbol were added to the blocks of D.

These details of structure will now be mimiced to form 3-(2n+2,n+1,n-1) designs which are not self-complementary and whose existence implies the existence of 2-(2n+1,n,n-1) designs with multiple extensions. The method will be demonstrated by constructing non-self-complementary 3-(18,9,7) designs from which we deduce a 2-(17,8,7) design with 16 different extensions.

For a 3-(18,9,7) design the parameters are $v = 18$, $b = 68$, $k = 9$, $r = 34$, $\lambda_2 = 16$, $\lambda_3 = 7$. We start by constructing 36 DD* blocks then to these add different sets of H blocks. A block pair disjoint 2-(9,4,3) D is needed. There is just one such design up to an isomorphism (see fig 4). Flag-poles are added to the blocks to form the complementary design D* which is a 2-(9,5,5). Then the numbers in the blocks of D* which are not flag-poles are duplicated in the other type-face and finally the type-faces are interchanged to make further blocks. Thus the 36 DD* blocks of figure 5 are formed.

2	3	4	5		1	6	7	8
5	8	2	7		0	6	3	4
0	3	1	7		4	8	5	6
0	2	6	8		5	7	1	4
6	7	0	5		2	8	1	3
1	8	0	4		3	7	2	6
4	7	3	8		0	1	2	5
4	6	1	2		3	5	0	8
1	5	3	6		2	4	0	7

D: 2-(9,4,3)

$v = 9$
$b = 18$
$r = 8$
$k = 4$
$\lambda = 3$

0	2	3	4	5		0	1	6	7	8
1	5	8	2	7		1	0	6	3	4
2	0	3	1	7		2	4	8	5	6
3	0	2	6	8		3	5	7	1	4
4	6	7	0	5		4	2	8	1	3
5	1	8	0	4		5	3	7	2	6
6	4	7	3	8		6	0	1	2	5
7	4	6	1	2		7	3	5	0	8
8	1	5	3	6		8	2	4	0	7

D*: 2-(9,5,5)

$v^* = 9$
$b^* = 18$
$r^* = 10$
$k^* = 5$
$\lambda^* = 5$

Figure 4: A block pair disjoint 2-(9,4,3) design and its complement forming a 3 array.

0	2	*2*	3	*3*	4	*4*	5	*5*		*0*	1	*1*	6	*6*	7	*7*	8	*8*
0	2	*2*	3	*3*	4	*4*	5	*5*		0	1	*1*	6	*6*	7	*7*	8	*8*
1	5	*5*	8	*8*	2	*2*	7	*7*		*1*	0	*0*	6	*6*	3	*3*	4	*4*
1	5	*5*	8	*8*	2	*2*	7	*7*		1	0	*0*	6	*6*	3	*3*	4	*4*
2	0	*0*	3	*3*	1	*1*	7	*7*		*2*	4	*4*	8	*8*	5	*5*	6	*6*
2	0	*0*	3	*3*	1	*1*	7	*7*		2	4	*4*	8	*8*	5	*5*	6	*6*
3	0	*0*	2	*2*	6	*6*	8	*8*		*3*	5	*5*	7	*7*	1	*1*	4	*4*
3	0	*0*	2	*2*	6	*6*	8	*8*		3	5	*5*	7	*7*	1	*1*	4	*4*
4	6	*6*	7	*7*	0	*0*	5	*5*		*4*	2	*2*	8	*8*	1	*1*	3	*3*
4	6	*6*	7	*7*	0	*0*	5	*5*		4	2	*2*	8	*8*	1	*1*	3	*3*
5	1	*1*	8	*8*	0	*0*	4	*4*		*5*	3	*3*	7	*7*	2	*2*	6	*6*
5	1	*1*	8	*8*	0	*0*	4	*4*		5	3	*3*	7	*7*	2	*2*	6	*6*
6	4	*4*	7	*7*	3	*3*	8	*8*		*6*	0	*0*	1	*1*	2	*2*	5	*5*
6	4	*4*	7	*7*	3	*3*	8	*8*		6	0	*0*	1	*1*	2	*2*	5	*5*
7	4	*4*	6	*6*	1	*1*	2	*2*		*7*	3	*3*	5	*5*	0	*0*	8	*8*
7	4	*4*	6	*6*	1	*1*	2	*2*		7	3	*3*	5	*5*	0	*0*	8	*8*
8	1	*1*	5	*5*	3	*3*	6	*6*		*8*	2	*2*	4	*4*	0	*0*	7	*7*
8	1	*1*	5	*5*	3	*3*	6	*6*		8	2	*2*	4	*4*	0	*0*	7	*7*

Figure 5: 36DD* blocks for a 3-(18,9,7) design.

It is time to pause and count triples. Deleting a type-face from figure 5 produces the 3-array of figure 4 which but for a missing symbol would be a 3-design obtained as an extension by complementation of a 2-(9,4,3) design. Therefore triples $\mathbf{xyz}$ occur 3 times in the DD* blocks. Since ($\mathbf{x}x$) is an automorphism for these blocks all triples containing three different numbers appear 3 times. The only other triples are those containing a repeated number, e.g. $x\mathbf{xy}$. These occur as often as the pair $\mathbf{xy}$ provided the $\mathbf{x}$ is not a flag-pole. From figure 4 the pair $\mathbf{xy}$ occurs 8 times but in just one of these $\mathbf{x}$ is a flag-pole. Therefore in the DD* blocks triples $x\mathbf{xy}$ occur 7 times. Since for the whole design $\lambda_3 = 7$ the pair $x\mathbf{x}$ cannot occur again and the remaining 32 blocks must be H blocks.

These 32 blocks fall into two classes of 16 according as they contain 0 or $\mathbf{0}$. Call these classes H(0) and H($\mathbf{0}$). No block in either class can contain a repeated number. Such triples that occur in the H blocks must do so 4 times. The trick is to make sure each such triple occurs twice in each of H(0) and H($\mathbf{0}$) with the proviso that triples containing 0 appear four times in H(0) and similarly for those containing $\mathbf{0}$ which are in H($\mathbf{0}$).

Since ($x\mathbf{x}$) is an automorphism for the DD* blocks the first H(0) block can be taken to be 012345678. To keep pairs including 0 balanced between the two type-faces take the last H(0) block to be $0\mathbf{12345678}$ (see fig 6). In the remaining 14 blocks of H(0) triples $0yz$ must occur 3 times and triples xyz not containing 0 must occur once. These observations suggest that excluding 0 and concentrating on one type-face we should have 3-design in these 14 blocks on the symbols *12345678*. But there *is* a 3-(8,4,1) design, which is self-complementary being the extension of a Hadamard 2-(7,3,1) design (see fig 6).

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*								
0	*1*	*2*	*3*	*4*	**5**	**6**	**7**	**8**	*1*	*2*	*3*	*4*				
0	**1**	**2**	**3**	**4**	*5*	*6*	*7*	*8*					*5*	*6*	*7*	*8*
0	*1*	*2*	**3**	**4**	**5**	**6**	*7*	*8*	*1*	*2*					*7*	*8*
0	**1**	**2**	*3*	*4*	*5*	*6*	**7**	**8**			*3*	*4*	*5*	*6*		
0	*1*	**2**	*3*	**4**	**5**	*6*	**7**	*8*	*1*		*3*			*6*		*8*
0	**1**	*2*	**3**	*4*	*5*	**6**	*7*	**8**		*2*		*4*	*5*		*7*	
0	*1*	**2**	**3**	*4*	**5**	*6*	*7*	**8**	*1*			*4*		*6*	*7*	
0	**1**	*2*	*3*	**4**	*5*	**6**	**7**	*8*		*2*	*3*		*5*			*8*
0	*1*	*2*	**3**	**4**	*5*	*6*	**7**	**8**	*1*	*2*			*5*	*6*		
0	**1**	**2**	*3*	*4*	**5**	**6**	*7*	*8*			*3*	*4*			*7*	*8*
0	*1*	**2**	*3*	**4**	*5*	**6**	*7*	**8**	*1*		*3*		*5*		*7*	
0	**1**	*2*	**3**	*4*	**5**	*6*	**7**	*8*		*2*		*4*		*6*		*8*
0	*1*	**2**	**3**	*4*	*5*	**6**	**7**	*8*	*1*			*4*	*5*			*8*
0	**1**	*2*	*3*	**4**	**5**	*6*	*7*	**8**		*2*	*3*			*6*	*7*	
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**								

H(*0*) Blocks

3-(8,4,1): $v = 8$, $k = 4$, $b = 14$, $r = 7$, $\lambda_3 = 3$, $\lambda_3 = 1$.

Figure 6: A set of H(*0*) blocks for a 3-(18,9,7) design.

The complement of the 3-(8,4,1) design is the same design so when the 14 blocks are completed with symbols from **12345678** the triple count for that type-face is correct as well. It remains to check triples containing both type-faces. Any pair of non-zero numbers appears 16 times (once for each block). Of these 4 are *xy*, 4 are **xy**, 4 are **x***y* and 4 are *x***y**. Thus all triples containing *0* appear 4 times in the H(*0*) blocks. Now take a triple *xyz* not containing *0*. This occurs in two blocks. The pair *xy* occurs without *z* in 1 + 3 - 2 = 2 blocks. Therefore *xy***z** occurs just twice in the H(*0*) blocks. Thus the H(*0*) blocks have the correct triple count.

To complete the 3-(18,9,7) design a set of 16 H(**0**) blocks is needed. One way of making these is to replace *0* by **0** in the H(*0*) blocks (see fig 7(i)). This produces a self-complementary 3-design. To obtain a non-self-complementary design interchange the type-faces down the 1-column of the H(**0**) blocks (see fig 7(ii)). In effect this amounts to making H blocks with respect to **0** and the two sets of symbols **1***2345678* and *1***2345678** so the triple count must be correct.

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0	*1*	*2*	*3*	*4*	**5**	**6**	**7**	**8**
0	**1**	**2**	**3**	**4**	*5*	*6*	*7*	*8*
0	*1*	*2*	**3**	**4**	**5**	**6**	*7*	*8*
0	**1**	**2**	*3*	*4*	*5*	*6*	**7**	**8**
0	*1*	**2**	*3*	**4**	**5**	*6*	**7**	*8*
0	**1**	*2*	**3**	*4*	*5*	**6**	*7*	**8**
0	*1*	**2**	**3**	*4*	**5**	*6*	*7*	**8**
0	**1**	*2*	*3*	**4**	*5*	**6**	**7**	*8*
0	*1*	*2*	**3**	**4**	*5*	*6*	**7**	**8**
0	**1**	**2**	*3*	*4*	**5**	**6**	*7*	*8*
0	*1*	**2**	*3*	**4**	*5*	**6**	*7*	**8**
0	**1**	*2*	**3**	*4*	**5**	*6*	**7**	*8*
0	*1*	**2**	**3**	*4*	*5*	**6**	**7**	*8*
0	**1**	*2*	*3*	**4**	**5**	*6*	*7*	**8**
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

(i) A set of H(**0**) blocks to complete to a self-complementary 3-(18,9,7) design.

0	**1**	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0	**1**	*2*	*3*	*4*	**5**	**6**	**7**	**8**
0	*1*	**2**	**3**	**4**	*5*	*6*	*7*	*8*
0	**1**	*2*	**3**	**4**	**5**	**6**	*7*	*8*
0	*1*	**2**	*3*	*4*	*5*	*6*	**7**	**8**
0	**1**	**2**	*3*	**4**	**5**	*6*	**7**	*8*
0	*1*	*2*	**3**	*4*	*5*	**6**	*7*	**8**
0	**1**	**2**	**3**	*4*	**5**	*6*	*7*	**8**
0	*1*	*2*	*3*	**4**	*5*	**6**	**7**	*8*
0	**1**	*2*	**3**	**4**	*5*	*6*	**7**	**8**
0	*1*	**2**	*3*	*4*	**5**	**6**	*7*	*8*
0	**1**	**2**	*3*	**4**	*5*	**6**	*7*	**8**
0	*1*	*2*	**3**	*4*	**5**	*6*	**7**	*8*
0	**1**	**2**	**3**	*4*	*5*	**6**	**7**	*8*
0	*1*	*2*	*3*	**4**	**5**	*6*	*7*	**8**
0	*1*	**2**	**3**	**4**	**5**	**6**	**7**	**8**

(ii) A set of H(**0**) blocks to complete to a non-self-complementary 3-(18,9,7) design.

<u>Figure 7</u>: Sample sets of H(**0**) blocks for making 3-(18,9,7) designs.

If a restriction on *0* is taken in these completed 3-designs then a 2-(17,8,7) design with at least 2 extensions is obtained. In fact it has at least 16 extensions because type-face changes on the columns of the H(**0**) can be made in many different ways. However isomorphic sets of blocks occur under these changes and a careful investigation is needed. To generate the 16 sets of H(**0**) blocks write out the pattern of type-faces in the H(*0*) blocks as in figure 8. Then interchange the types in the 0-column and any other one or two columns. (Interchanging down more columns only produces arrays isomorphic to those already obtained.)

0	1	2	3	4	5	6	7	8	(Column numbers)
1	1	1	1	1	1	1	1	1	
1	1	1	1	1	.	.	.	.	
1	.	.	.	.	1	1	1	1	
1	1	1	.	.	.		1	1	
1	.	.	1	1	1	1	.	.	
1	1	.	1	.	.	1	.	1	
1	.	1	.	1	1	.	1	.	
1	1	.	.	1	.	1	1	.	
1	.	1	1	.	1	.	.	1	
1	1	1	.	.	1	1	.	.	
1	.	.	1	1	.	.	1	1	
1	1	.	1	.	1	.	1	.	
1	.	1	.	1	.	1	.	1	
1	1	.	.	1	1	.	.	1	
1	.	1	1	.	.	1	1	.	
1	.	.	.	.	.	.	.	.	

Change 0-column and:

No other column	1 way,	
1 other column	8 ways	
2 other columns	7 ways	
	16	

16 non-isomorphic patterns

Equivalent column pair changes:

1,2 - 3,4 - 5,6 - 7,8
1,3 - 2,4 - 5,7 - 6,8
1,4 - 2,3 - 6,7 - 5,8
1,5 - 2,6 - 3,7 - 4,8
1,6 - 2,5 - 3,8 - 4,7
1,7 - 2,8 - 3,5 - 4,6
1,8 - 2,7 - 3,6 - 4,5 .

<u>Figure 8</u>: Scheme for generating H(**0**) blocks from a set of H(*0*) blocks.

There are 8 ways of changing one column other than the 0-column. There are 7 ways of changing two columns since for example changing columns 1 and 2 produces the same blocks as does changing columns 3 and 4 or 5 and 6 or 7 and 8. The 16th way comes from not changing any column other than the 0-column. Thus a 2-(17,8,7) design with sixteen extensions can be constructed.

The method generalises quite readily. Thus a block pair disjoint 2-(17,8,7) design exists and Hadamard matrices of order 16 exist so it is possible to construct 3-(34,17,15) designs which are not self-complementary.

There are infinite families of block pair disjoint 2-(2n+1,n,n-1) designs. For example if 2n+1 is a prime power, put non-zero squares in the corresponding finite field into one block, put the non-zero non-squares into another block and use the algebra of the field to generate more blocks. The details are given Hall [2], p.209. Hadamard matrices and therefore Hadamard designs also exist in infinite families. If it were known that block pair disjoint designs exist for all n or that Hadamard martices exist for all possible orders then it would be possible to assert that there exist 2-designs with more than any specified number of extensions.

For block pair disjoint 2-(2n+1,n,n-1) designs with $2n+1 \equiv 3 \pmod 4$ the H blocks contain triples an odd number of times and 3-(4n+2,2n+1,2n-1) designs which are not self-complementary are much harder to construct.

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